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New Improved Exact Sequences of Witt Groups

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Two exact sequences of Witt groups are constructed, extending ones obtained earlier by the author. They involve Witt groups of forms over a base field K, quadratic extension L, and quaternion division algebra D containing L as a maximal subfield. The mappings in these sequences arise by use of the tensor product for "going up" in the inclusions $K \subset L \subset D$ and by use of trace maps for "going down." The sequences exhibit a pleasing degree of symmetry and yield results on the relative sizes of the Witt groups. Also a result on the relative sizes of the groups of square classes of K and L is obtained.

We obtain two new exact sequences of Witt groups of forms, extending the sequences appearing in [2]. The new sequences exhibit a pleasing degree of symmetry and also enable us to deduce some new results concerning the size of Witt groups and of square classes of fields.

Our notation and terminology will be as in [2]. Let K be a field (characteristic $\neq 2$), and L a quadratic extension of K. We write $L = K(\sqrt{a})$ for some $a \in K$. Let D be the quaternion division algebra $(a, b/K)$ generated by i, j satisfying $i^2 = a$, $j^2 = b$, $ij = -ji$, etc.,—and $\hat{}$ denote the involutions on D given by $\bar{i} = -i$, $\bar{j} = -j$, and $\hat{i} = -i$, $\hat{j} = j$, respectively. We write $W(K)$, $W(L)$, $W(L, -)$, $W(D, -)$, $W(D, \hat{})$ for the Witt group of nonsingular forms which are quadratic over K, quadratic over L, hermitian over L w.r.t.–, hermitian over D w.r.t.–, and hermitian over D w.r.t. $\hat{}$, respectively. (Forms which are hermitian over $(D, \hat{})$ may equivalently be viewed as skew-hermitian over $(D, -)$ [2].)

PROPOSITION 1. *There exists an exact sequence of groups (in fact of $W(K)$-modules)*

$$0 \longrightarrow W(L, -) \xrightarrow{T_1^*} W(K) \xrightarrow{U_1} W(L) \xrightarrow{T_2^*} W(K)$$

$$\xrightarrow{U_2} W(L, -) \longrightarrow 0,$$

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where T_i^*, $i=1,2$ *is induced by* T_i: $L \to K$, $T_i(r_1 + r_2\sqrt{a}) = r_i$, *each* $r_i \in K$ *and* U_i, $i=1,2$, *is induced by the tensor product. i.e., if* ψ: $V \times V \to K$ *represents an element of* $W(K)$, *then* $U_i(\psi)$ *is the form* $(V \otimes_K L) \times (V \otimes_K L) \to L$ *given by* $x \otimes \alpha$, $y \otimes \beta \mapsto \psi(x,y)\alpha\beta$ *for* $i=1$, *and* $\psi(x,y)\bar{\alpha}\beta$ *for* $i=2$.

Proof. We proved exactness as far as the second $W(K)$ term in [2]. To see that $U_2 \circ T_2^*$ is zero consider a one-dimensional form $\langle r + s\sqrt{a}\rangle$ representing an element of $W(L)$. It is easily checked that, under T_2^*, this maps to the two-dimensional form over K with matrix $\left(\begin{smallmatrix} s & r \\ r & as \end{smallmatrix}\right)$, and under U_2, we obtain the same matrix representing a form over $(L, -)$. This matrix has determinant $as^2 - r^2 = -z\bar{z}$, where $z = r + s\sqrt{a}$ and hence represents a hyperbolic form. Since any form is an orthogonal sum of one-dimensional forms, it follows that $U_2 \circ T_2^*$ is the zero map.

Conversely suppose $U_2(\psi)$ is hyperbolic, ψ: $V \times V \to K$ giving an element of $W(K)$. Then there exists an element $v \otimes 1 + w \otimes \sqrt{a} \in (V \otimes_K L)$ with $v, w \in V$ for which $U_2(\psi)$ represents zero. This implies that $\psi(v,v) - a\psi(w,w) = 0$. Thus v, w span a two-dimensional subspace of V with ψ/V having matrix of the form $\left(\begin{smallmatrix} s & r \\ r & as \end{smallmatrix}\right)$. We can clearly decompose ψ into an orthogonal sum of such forms and so ψ is in the image of T_2^*.

Exactness at the final $W(L, -)$ stage is immediate since U_2 is clearly surjective. (Any form over $(L, -)$ has diagonalization consisting entirely of elements of K.)

COROLLARY.

$$|W(K)|^2 = |W(L)|\,|W(L,-)|^2$$

where $|\ |$ *denotes cardinality.*

EXAMPLE. Let K be a p-adic field, $p \neq 2$. Then it is well known [1, p. 149] that $|W(K)| = 16$, $|W(L)| = 16$. Also $|W(L,-)| = 4$, see [3, Appendix 2], and so our formula is verified in this case.

PROPOSITION 2. *There is an exact sequence*

$$0 \longrightarrow W(D,-) \xrightarrow{T_1^*} W(L,-) \xrightarrow{U_1} W(D,\hat{}\,) \xrightarrow{T_2^*} W(L) \xrightarrow{U_2} W(D,\hat{}\,)$$
$$\xrightarrow{T_1^*} W(L,-) \xrightarrow{U_3} W(D,-) \longrightarrow 0,$$

where T_i^*, $i=1,2$, *is induced by* T_i: $D \to L$, $T_i(z_1 + z_2 j) = z_i$, *each* $z_i \in L$

and U_i, $i = 1, 2, 3$ *is induced by the tensor product. (The definition of* U_1 *and* U_3 *is clear but that of* U_2 *needs a bit more care. For* $\psi\colon V \times V \to L$ *representing an element of* $W(L)$ *we define* $U_2(\psi)\colon (V \otimes_L D) \times (V \otimes_L D) \to D$,

$$x \otimes \lambda,\ y \otimes \mu \mapsto \hat{\lambda}\psi(x, y)\, k\mu,$$

where $k = ij$.*)*

Proof. Exactness up to the $W(L)$ term is done in [2]. To see $U_2 \circ T_2^*$ is zero take $z_1 + z_2 j$, $z_i \in L$, $i = 1, 2$, a one-dimensional form over $(D, \hat{\ })$. This gives the form over L with matrix $\left(\begin{smallmatrix} z_2 & z_1 \\ z_1 & b\bar{z}_2 \end{smallmatrix}\right)$ and applying U_2 we get $\left(\begin{smallmatrix} z_2 k & z_1 k \\ z_1 k & b\bar{z}_2 k \end{smallmatrix}\right)$. This has diagonalization $\langle z_2 k, b\bar{z}_2 k - (z_1^2/z_2)k \rangle$ as a form over $(D, \hat{\ })$. (If $z_2 = 0$, then the form is already hyperbolic over L before applying U_2.) Now $\hat{q} z_2 k q = (z_1^2/z_2)k - b\bar{z}_2 k$, where $q = z_1/z_2 + j$ and thus our form is hyperbolic over $(D, \hat{\ })$, i.e., $U_2 \circ T_2^* = 0$. (Note—we cannot take the determinant to prove a two-dimensional form over D to be hyperbolic!) Conversely given ψ in $W(L)$ such that $U_2(\psi)$ is hyperbolic it is easily shown by the same technique as before that ψ can be written as a sum of two-dimensional forms like $\left(\begin{smallmatrix} z_2 & z_1 \\ z_1 & b\bar{z}_2 \end{smallmatrix}\right)$. Thus we have exactness of $W(L)$.

Next it is clear that $T_1^* \circ U_2$ is zero. Conversely if $T_1^* \phi$ is hyperbolic over $(L, -)$ for $\phi \in W(D, \hat{\ })$, then $T_1 \phi(x, x) = 0$ for some x and so $\phi(x, x) = zj$ for some $z \in L$. Thus

$$\phi(x, x) = \frac{zi}{a} k.$$

Hence ϕ can be seen to be in the image of U_2. Exactness at the second $W(L)$ term can be shown by similar methods and U_3 is clearly surjective, thus completing the proof.

COROLLARY.

$$|W(D, -)|\, |W(D, \hat{\ })| = |W(K)|.$$

Proof. Use Proposition 2 together with the corollary to Proposition 1.

EXAMPLE. For K p-adic, $p \neq 2$, there is a unique quaternion division algebra D. Now $|W(D, -)| = 2$, $|W(D, \hat{\ })| = 8$, see [4], and $|W(K)| = 16$ as mentioned before.

Write $\dot{K}/\dot{K}^2$, $\dot{L}/\dot{L}^2$ for the group of square classes of K, L, respectively, and $\dot{K}/L\bar{L}$ for the classes of K modulo norms from L. Proposition 1 suggests the following:

PROPOSITION 3. *There is an exact sequence of multiplicative groups*

$$1 \to \{\dot{K}, a\dot{K}^2\} \to \frac{\dot{K}}{\dot{K}^2} \to \frac{\dot{L}}{\dot{L}^2} \xrightarrow{N} \frac{\dot{K}}{\dot{K}^2} \to \frac{\dot{K}}{L\bar{L}} \to 1,$$

N being the norm map from L to K, the other maps being the obvious ones.

Proof. Most of this sequence (i.e., up to the second $\dot{K}/\dot{K}^2$) appears in [1, p. 202]. Exactness at each point can be deduced either by using Proposition 1 or independently by using Hilbert's theorem 90 to show exactness at $\dot{L}/\dot{L}^2$, exactness at other points in the sequence being almost immediate.

COROLLARY.

$$\left|\frac{\dot{L}}{\dot{L}^2}\right| \left|\frac{\dot{K}}{L\bar{L}}\right| = \frac{1}{2}\left|\frac{\dot{K}}{\dot{K}^2}\right|^2.$$

Note. This improves the inequality

$$\frac{1}{2}\left|\frac{\dot{K}}{\dot{K}^2}\right| \leqslant \left|\frac{\dot{L}}{\dot{L}^2}\right| \leqslant \frac{1}{2}\left|\frac{\dot{K}}{\dot{K}^2}\right|^2$$

appearing in [1, p. 203] since $|\dot{K}/L\bar{L}| \leqslant |\dot{K}/\dot{K}^2|$.

Comment. It might be hoped that, in a similar fashion to the above, Proposition 2 might yield another exact sequence involving square classes, etc. There is a problem about what would correspond to $W(D, \hat{\ })$. One possible choice is the set of isometry classes of one-dimensional forms over $(D, \hat{\ })$. Unfortunately this set has no group structure and even though some notion of exactness remains (since this set has a distinguished element 1), the existence and usefulness of such a sequence is in doubt. Alternatively we could take $\dot{K}/D\hat{D}$, i.e., elements of K modulo $w\hat{w}$, $w \in D$, which is a group. Mimicking Proposition 3, since the norm map $K/D\hat{D} \to L/L^2$ is trivial, we have a *short* exact sequence

$$1 \to \{L\bar{L}, bL\bar{L}\} \to \frac{\dot{K}}{L\bar{L}} \to \frac{\dot{K}}{D\hat{D}} \to 1$$

yielding the result that $|\dot{K}/L\bar{L}| = 2\,|\dot{K}/D\hat{D}|$. This can, of course, be obtained by an easy calculation without mention of Witt groups.

References

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